Integral Expressions for Component Pressure Losses in Wall Flow Filters

(…or Where did that $\frac{2}{3}$ come from?)

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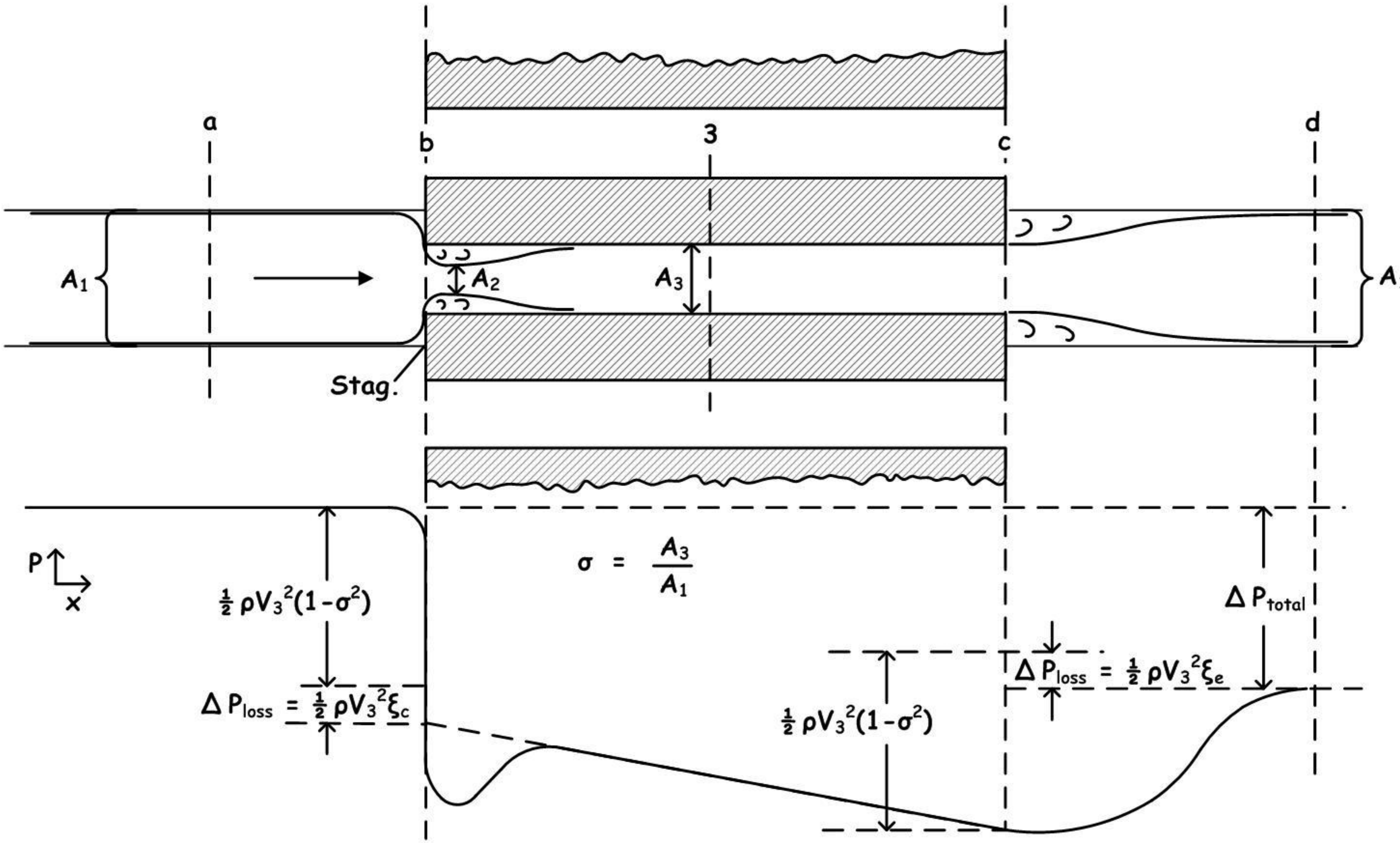
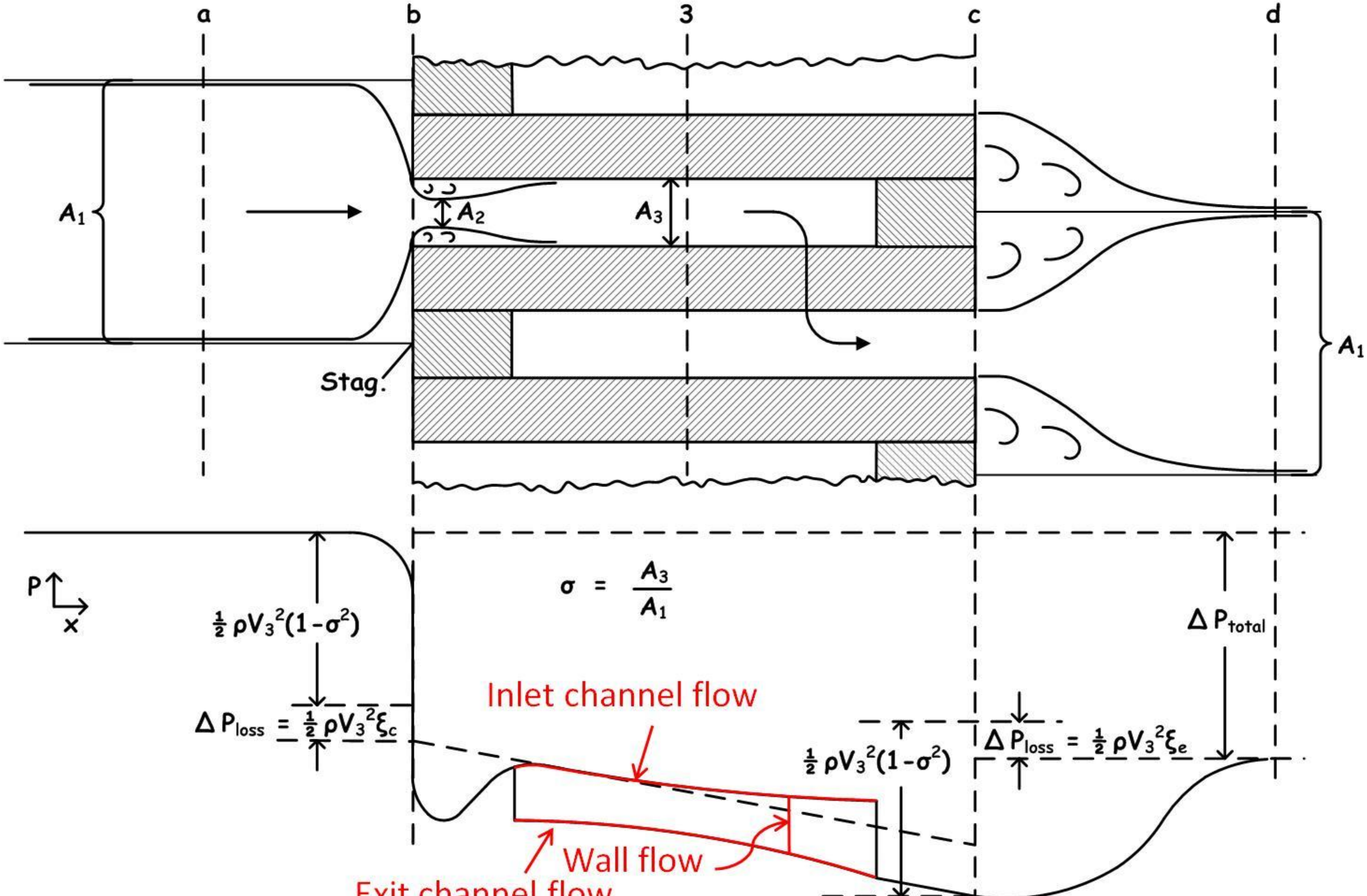
Pressure changes for flow through a multichannel device



Kays, W. M., Trans. ASME, Vol. 72, 1950

Pressure changes for flow through a wall-flow filter



Basic pressure drop expression

$$\Delta P = \underbrace{\frac{\mu u_0 \alpha w}{4kL}}_{\text{Wall flow resistance}} + \underbrace{\overset{?}{\frac{2}{3}} \frac{\mu F u_0 L}{\alpha^2}}_{\text{Channel friction resistance}}$$

Darcy

Poiseuille

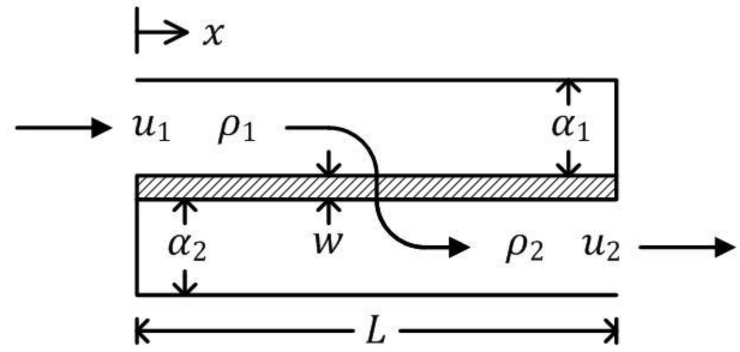
Konstandopoulos, A. G. and Johnson, J. H., SAE 890405
Konstandopoulos, A. G. et al., SAE 1999-01-0468

Assumptions:

- Uniform wall flow
- Incompressible
- Neglect inertial contributions
  - Expansion/contraction loss
  - Forchheimer correction to Darcy's law
- Neglect effect of plug length
- Square channels
- Symmetric inlet/outlet channel size

ΔP	= Pressure drop (Pa)
α	= Channel hydraulic diameter (m)
k	= Wall permeability (m^2)
L	= Filter length (m)
μ	= Fluid shear viscosity (kg/m/s)
u_0	= Fluid velocity at entrance (m/s)
w	= Filter wall thickness (m)
F	$= 2\mathrm{Po} = 28.454$ (For laminar flow in square channels)

Fundamental governing equations



Mass balance:

$$\rho_1 \frac{du_1}{dx} + \frac{n_w \rho_1 u_{w,1}}{\alpha_1} = 0$$

$$\rho_2 \frac{du_2}{dx} - \frac{n_w \rho_2 u_{w,2}}{\alpha_2} = 0$$

Momentum balance:

$$\rho_1 \frac{d(u_1^2)}{dx} + \frac{dP_1}{dx} + F\frac{\mu u_1}{\alpha_1^2} = 0$$

$$\rho_2 \frac{d(u_2^2)}{dx} + \frac{dP_2}{dx} + F\frac{\mu u_2}{\alpha_2^2} = 0$$

Bissett, E. J., Chem. Eng. Sci., 39, 1984
Konstandopoulos, A. G. et al., SAE 1999-01-0468

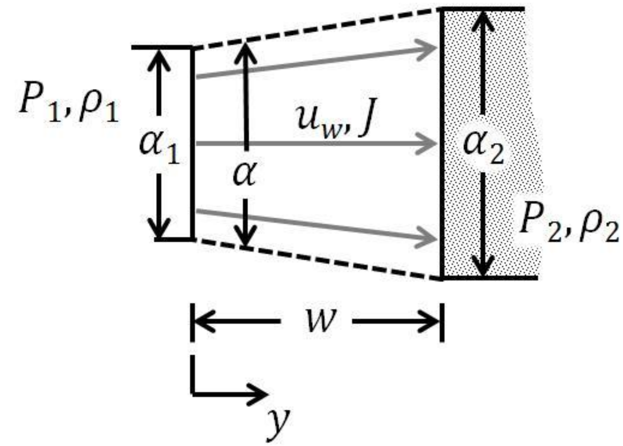
Simplifying approximations:

$$\rho_1 = \frac{P_1(x=0)}{R\,T} \qquad \rho_2 = \frac{P_2(x=L)}{R\,T}$$

P = Pressure (Pa)
T = Temperature (K)
R = Ideal gas constant (J/kg/K)
ρ = Fluid density (kg/m^3)
u_w = Wall flow velocity (m/s)
n_w = Average number of permeable walls per channel

Darcy's law through the wall

$$\boxed{\frac{dP}{dy} = -\frac{\mu}{k}u_w - \beta\rho u_w^2}$$



$$\alpha(y) = \alpha_1 + \frac{y}{w}(\alpha_2 - \alpha_1)$$

Ideal gas: $\rho = \dfrac{P}{RT}$

Mass flux: $J = \rho u_w$

$$\frac{dP}{dy} = -\frac{\mu}{k}\frac{RT}{P}J - \beta\frac{RT}{P}J^2$$

Define mean values:

$$\bar{J} = J\left(\frac{w}{2}\right)$$

$$\bar{\alpha} = \alpha\left(\frac{w}{2}\right) = \frac{1}{2}(\alpha_1 + \alpha_2)$$

By mass conservation: $J(y) = \dfrac{\bar{J}\bar{\alpha}}{\alpha(y)}$

k	= Permeability (m^2)
β	= Forchheimer coef. (1/m)

Darcy's law through the wall (cont.)

$$\int_{P_1}^{P_2} P\,dP = -\frac{\mu}{k} RT\bar{J}w \int_0^w \frac{\overline{\alpha}}{w\alpha(y)}\,dy - \beta RT\bar{J}^2 w \int_0^w \frac{\overline{\alpha}^2}{w\alpha(y)^2}\,dy$$

Integrate left hand side:

$$\int_{P_1}^{P_2} P\,dP = \tfrac{1}{2}(P_2^2 - P_1^2) = \tfrac{1}{2}[P_2^2 - (P_2 + \Delta P_w)^2] = -\tfrac{1}{2}\Delta P_w(2P_2 + \Delta P_w)$$

Define shape factors:

$$\phi_k = \int_0^w \frac{\overline{\alpha}}{w\alpha(y)}\,dy = \begin{cases} \dfrac{\overline{\alpha}}{(\alpha_1 - \alpha_2)} \ln\left(\dfrac{\alpha_1}{\alpha_2}\right) & \text{if } \alpha_1 \neq \alpha_2 \\ 1 & \text{if } \alpha_1 = \alpha_2 \end{cases}$$

$$\phi_\beta = \int_0^w \frac{\overline{\alpha}^2}{w\alpha(y)^2}\,dy = \frac{\overline{\alpha}^2}{\alpha_1\alpha_2}$$

Darcy's law through the wall (cont.)

$$\tfrac{1}{2}\Delta P_w(2P_2 + \Delta P_w) = \frac{\mu}{k} RT\bar{J}w\phi_k + \beta RT\bar{J}^2 w\phi_\beta$$

Define some more mean values:

$$\bar{\rho} = \frac{\rho_1 + \rho_2}{2} = \frac{P_1 + P_2}{2RT} = \frac{(P_2 + \Delta P_w) + P_2}{2RT} = \frac{2P_2 + \Delta P_w}{2RT}$$

$$\bar{u}_w = \frac{\bar{J}}{\bar{\rho}}$$

By substitution:

$$\Delta P_w \cancel{\bar{\rho} RT} = \frac{\mu}{k} \cancel{\bar{\rho} RT} \bar{u}_w w\phi_k + \beta \bar{\rho}^{\cancel{2}} \cancel{RT} \bar{u}_w^2 w\phi_\beta$$

$$\boxed{\Delta P_w = \frac{\mu}{k} \bar{u}_w w\phi_k + \beta \bar{\rho} \bar{u}_w^2 w\phi_\beta}$$

Definition of characteristic and dimensionless parameters

$$P^* = \frac{\mu^2}{\overline{\rho}\,\overline{\alpha}^2} \qquad \text{(Pa)}$$

$$\overline{u}_0 = \frac{\dot{m}}{\overline{\rho}\,\overline{\alpha}^2} \qquad \text{(m/s)}$$

$$\hat{x} = \frac{x}{L}$$

$$\hat{P} = \frac{P}{P^*}$$

$$\hat{u}_1 = \frac{u_1}{\overline{u}_0}\hat{\rho}_1\hat{\alpha}_1^2$$

$$\mathrm{Re} = \frac{\overline{u}_0\,\overline{\rho}\,\overline{\alpha}}{\mu}$$

$$\hat{\rho}_1 = \frac{\rho_1}{\overline{\rho}} \qquad \hat{\rho}_2 = \frac{\rho_2}{\overline{\rho}}$$

$$\hat{u}_2 = \frac{u_2}{\overline{u}_0}\hat{\rho}_2\hat{\alpha}_2^2$$

$$\mathcal{F} = \frac{2\,\mathrm{Po}\,L}{\overline{\alpha}}$$

$$\hat{\alpha}_1 = \frac{\alpha_1}{\overline{\alpha}} \qquad \hat{\alpha}_2 = \frac{\alpha_2}{\overline{\alpha}}$$

$$\hat{u}_w = \frac{\overline{u}_w}{\overline{u}_0}\frac{n_w L}{\overline{\alpha}}$$

$$\mathcal{K} = \frac{\overline{\alpha}^2 w}{n_w k L}\phi_k$$

$$\hat{u}_1(0) = \hat{u}_2(1) = 1$$

$$\hat{u}_1(1) = \hat{u}_2(0) = 0$$

$$0 \le \hat{x} \le 1$$

$$\mathcal{B} = \frac{\beta\overline{\alpha}^2 w}{n_w^2 L^2}\phi_\beta$$

$\dot{m}$ is the single-channel mass flow rate.

Dimensionless forms of governing equations

Mass balance:

$$\frac{d\hat{u}_1}{d\hat{x}} + \hat{u}_w = 0$$

$$\frac{d\hat{u}_2}{d\hat{x}} - \hat{u}_w = 0$$

Momentum balance:

$$\mathrm{Re}^2 \frac{d}{d\hat{x}}(\hat{u}_1^2) + \hat{\rho}_1 \hat{\alpha}_1^4 \frac{d\hat{P}_1}{d\hat{x}} + \mathcal{F}\,\mathrm{Re}\hat{u}_1 = 0$$

$$\mathrm{Re}^2 \frac{d}{d\hat{x}}(\hat{u}_2^2) + \hat{\rho}_2 \hat{\alpha}_2^4 \frac{d\hat{P}_2}{d\hat{x}} + \mathcal{F}\,\mathrm{Re}\hat{u}_2 = 0$$

Darcy's law through the wall:

$$\Delta\hat{P}_w = \mathcal{K}\mathrm{Re}\hat{u}_w + \mathcal{B}\mathrm{Re}^2\hat{u}_w^2$$

Basic pressure drop expression:

With assumptions:

$$\hat{\alpha}_1 = \hat{\alpha}_2 = 1$$

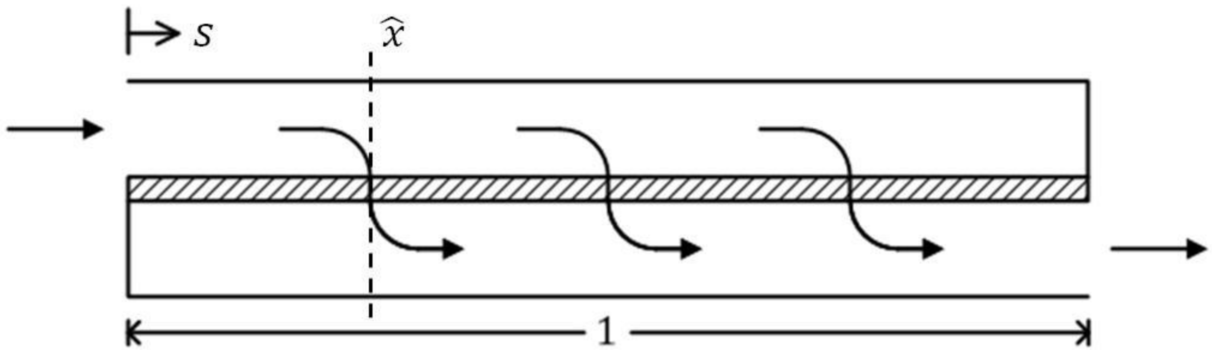
$$\hat{\rho}_1 = \hat{\rho}_2 = 1$$

$$\mathcal{B} = 0$$

$$\hat{u}_w(\hat{x}) = 1$$

$$\Delta\hat{P} = \mathcal{K}\mathrm{Re} + \tfrac{2}{3}\mathcal{F}\,\mathrm{Re}$$

Calculating component pressure changes



The filter has an infinite number of paths for the gas to take, with each one crossing the wall at a different $\widehat{x}$. Therefore each stream has a different $\Delta\widehat{P}_1$, $\Delta\widehat{P}_2$, and $\Delta\widehat{P}_w$ depending on where it crosses the wall. The total pressure drop for a stream crossing the wall at $\widehat{x}$ is:

$$\Delta\widehat{P}_s = \frac{\mathrm{Re}^2}{\widehat{\rho}_1\widehat{\alpha}_1^4}\left(\widehat{u}_1(\widehat{x})^2 - 1\right) + \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\widehat{\alpha}_1^4}\int_0^{\widehat{x}} \widehat{u}_1(s)\,ds$$

$$+ \mathcal{K}\mathrm{Re}\,\widehat{u}_w(\widehat{x}) + \mathcal{B}\mathrm{Re}^2\,\widehat{u}_w(\widehat{x})^2$$

$$+ \frac{\mathrm{Re}^2}{\widehat{\rho}_2\widehat{\alpha}_2^4}\left(1 - \widehat{u}_2(\widehat{x})^2\right) + \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\widehat{\alpha}_2^4}\int_{\widehat{x}}^{1} \widehat{u}_2(s)\,ds$$

Calculating component pressure changes (cont.)

We can divide the previous equation into four components:

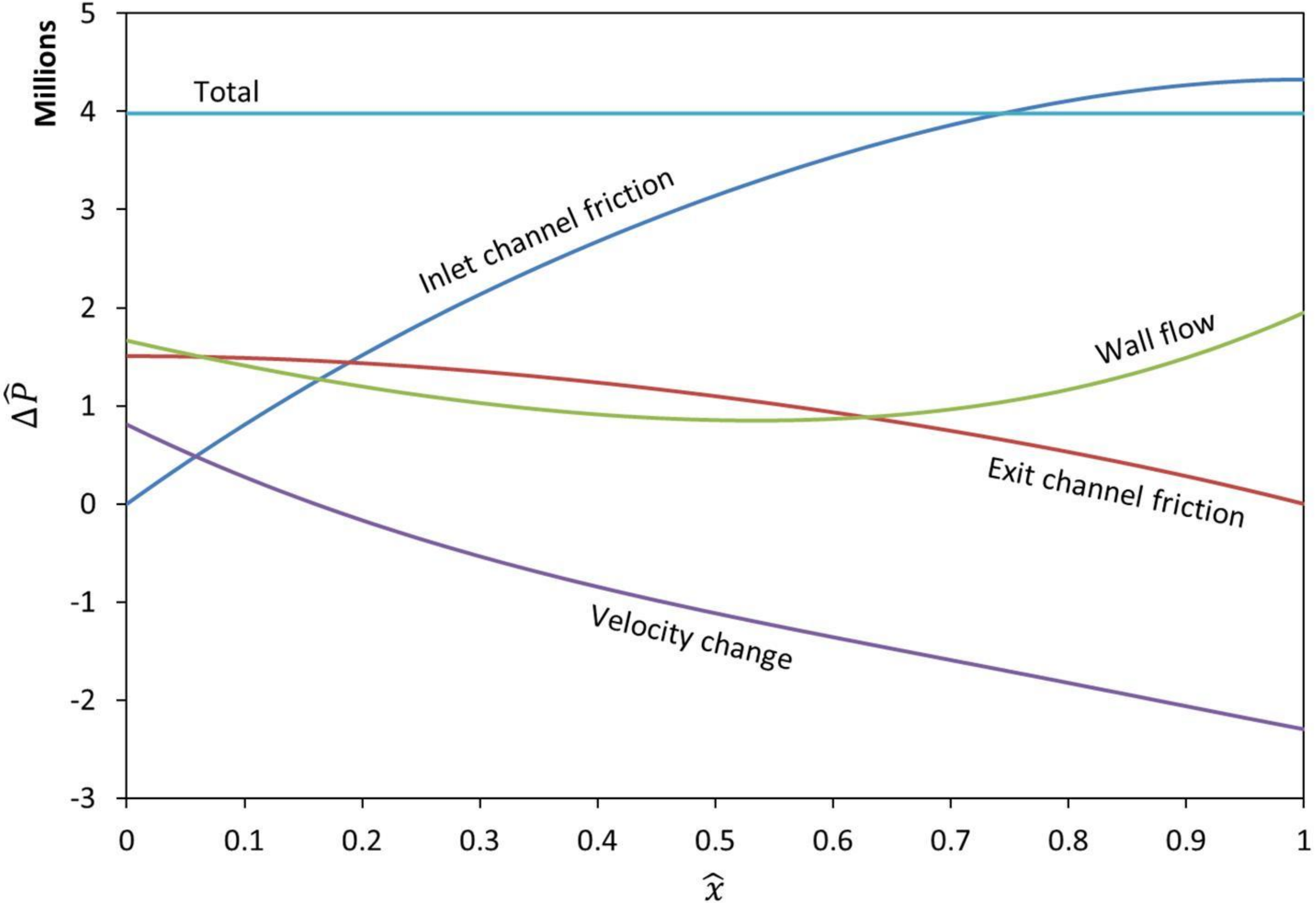
Inlet channel friction: $$\Delta\widehat{P}_{s,1}(\widehat{x}) = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\widehat{\alpha}_1^4}\int_0^{\widehat{x}}\widehat{u}_1(s)\,ds$$

Exit channel friction: $$\Delta\widehat{P}_{s,2}(\widehat{x}) = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\widehat{\alpha}_2^4}\int_{\widehat{x}}^{1}\widehat{u}_2(s)\,ds$$

Wall flow: $$\Delta\widehat{P}_{s,w}(\widehat{x}) = \mathcal{K}\mathrm{Re}\,\widehat{u}_w(\widehat{x}) + \mathcal{B}\mathrm{Re}^2\widehat{u}_w(\widehat{x})^2$$

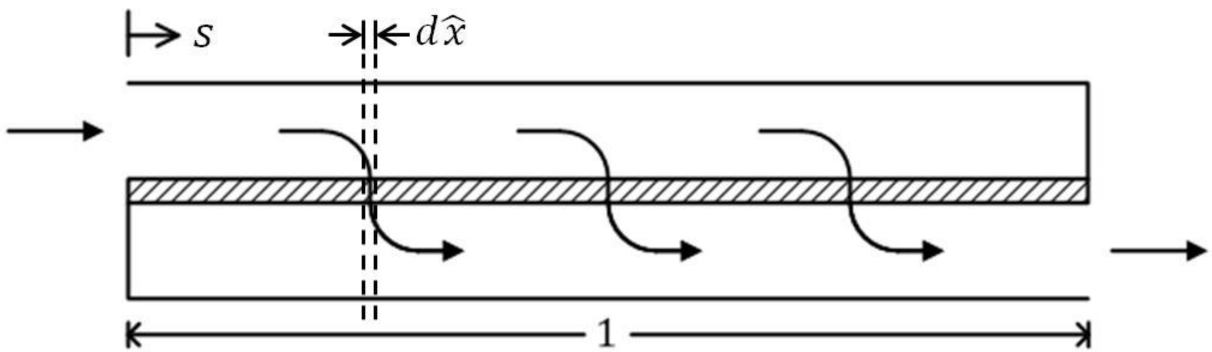
Velocity change: $$\Delta\widehat{P}_{s,u}(\widehat{x}) = \frac{\mathrm{Re}^2}{\widehat{\rho}_1\widehat{\alpha}_1^4}\left(\widehat{u}_1(\widehat{x})^2 - 1\right) + \frac{\mathrm{Re}^2}{\widehat{\rho}_2\widehat{\alpha}_2^4}\left(1 - \widehat{u}_2(\widehat{x})^2\right)$$

Component pressure changes as a function of wall crossing location



Calculating component pressure changes (cont.)

For each component, we want to compute the **mass average** over all exhaust paths.



If $\dot{m}$ is the total mass flow rate entering the inlet channel, then the mass flow rate of a stream crossing the wall at x is $d\dot{m}$, and the average dimensionless pressure drop over all streams is:

$$\Delta\overline{\hat{P}}_s = \frac{1}{\dot{m}}\int_s \Delta\hat{P}_s d\dot{m}$$

The mass flow rate through each stream is: $d\dot{m} = \bar{\rho}\bar{\alpha}n_w\bar{u}_w dx = \dot{m}\hat{u}_w d\hat{x}$

Thus:

$$\boxed{\Delta\overline{\hat{P}}_s = \int_0^1 \Delta\hat{P}_s \hat{u}_w d\hat{x}}$$

Calculating component pressure changes: Inlet channel friction

$$\Delta\overline{\widehat{P}}_{s,1} = \int_0^1 \left[\frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\widehat{\alpha}_1^4}\int_0^{\widehat{x}} \widehat{u}_1(s)\,ds\right]\widehat{u}_w(\widehat{x})\,d\widehat{x} = -\frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\widehat{\alpha}_1^4}\int_0^1 \int_0^{\widehat{x}} \widehat{u}_1(s)\,ds\left(\frac{d}{d\widehat{x}}\widehat{u}_1(\widehat{x})\right)d\widehat{x}$$

Recall that:

$$\widehat{u}_w = -\frac{d\widehat{u}_1}{d\widehat{x}}$$

By invoking the following two integral identities:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\frac{d}{dq}\int_p^q f(x)\,dx = f(q) \quad [p \text{ constant}]$$

$$\int_0^1 \underbrace{\int_0^{\widehat{x}} \widehat{u}_1(s)\,ds}_{u}\underbrace{\left(\frac{d}{d\widehat{x}}\widehat{u}_1(\widehat{x})\right)}_{v}d\widehat{x} = \widehat{u}_1(\widehat{x})\int_0^{\widehat{x}} \widehat{u}_1(s)\,ds\Bigg|_{\widehat{x}=0}^{\widehat{x}=1} - \int_0^1 \widehat{u}_1(\widehat{x})\frac{d}{d\underbrace{\widehat{x}}_{q}}\int_{\underbrace{0}_{p}}^{\widehat{x}} \underbrace{\widehat{u}_1(s)ds}_{f(x)dx}\,d\widehat{x}$$

$$= \underbrace{\cancel{\widehat{u}_1(1)}}_{=0}\int_0^1 \widehat{u}_1(s)\,ds - \widehat{u}_1(0)\underbrace{\cancel{\int_0^0 \widehat{u}_1(s)\,ds}}_{=0} - \int_0^1 \widehat{u}_1(\widehat{x})^2\,d\widehat{x}$$

$$\boxed{\Delta\overline{\widehat{P}}_{s,1} = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\widehat{\alpha}_1^4}\int_0^1 \widehat{u}_1(\widehat{x})^2\,d\widehat{x}}$$

Beyer, W. H., *CRC Standard Mathematical Tables*, 26th ed., CRC Press, 1981

Calculating component pressure changes: Exit channel friction

$$\Delta\overline{\widehat{P}}_{s,2} = \int_0^1 \left[\frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\widehat{\alpha}_2^4}\int_{\widehat{x}}^1 \widehat{u}_2(s)ds\right]\widehat{u}_w(\widehat{x})d\widehat{x} = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\widehat{\alpha}_2^4}\int_0^1\int_{\widehat{x}}^1 \widehat{u}_2(s)ds\left(\frac{d}{d\widehat{x}}\widehat{u}_2(\widehat{x})\right)d\widehat{x}$$

By invoking the following two integral identities:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\frac{d}{dp}\int_p^q f(x)dx = -f(p) \quad [q \text{ constant}]$$

Recall that:

$$\widehat{u}_w = \frac{d\widehat{u}_2}{d\widehat{x}}$$

$$\int_0^1 \underbrace{\int_{\widehat{x}}^1 \widehat{u}_2(s)ds}_{u}\underbrace{\left(\frac{d}{d\widehat{x}}\widehat{u}_2(\widehat{x})\right)}_{v}d\widehat{x} = \widehat{u}_2(\widehat{x})\int_{\widehat{x}}^1 \widehat{u}_2(s)ds\Bigg|_{\widehat{x}=0}^{\widehat{x}=1} - \int_0^1 \widehat{u}_2(\widehat{x})\underset{p}{\frac{d}{d\widehat{x}}}\int_{\widehat{x}}^{1 \leftarrow q} \underbrace{\widehat{u}_2(s)ds}_{f(x)dx}d\widehat{x}$$

$$= \widehat{u}_2(1)\underbrace{\cancel{\int_1^1 \widehat{u}_2(s)ds}}_{=0} - \underbrace{\cancel{\widehat{u}_2(0)}}_{=0}\int_0^1 \widehat{u}_2(s)ds + \int_0^1 \widehat{u}_2(\widehat{x})^2 d\widehat{x}$$

$$\boxed{\Delta\overline{\widehat{P}}_{s,2} = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\widehat{\alpha}_2^4}\int_0^1 \widehat{u}_2(\widehat{x})^2 d\widehat{x}}$$

Beyer, W. H., *CRC Standard Mathematical Tables*, 26th ed., CRC Press, 1981

Calculating component pressure changes: Velocity change

$$\Delta\overline{\widehat{P}}_{s,u} = \frac{\mathrm{Re}^2}{\widehat{\rho}_1 \widehat{\alpha}_1^4}\int_0^1 (\widehat{u}_1(\widehat{x})^2 - 1)\widehat{u}_w(\widehat{x})d\widehat{x} + \frac{\mathrm{Re}^2}{\widehat{\rho}_2 \widehat{\alpha}_2^4}\int_0^1 (1 - \widehat{u}_2(\widehat{x})^2)\,\widehat{u}_w(\widehat{x})d\widehat{x}$$

$$\boxed{\widehat{u}_w = -\frac{d\widehat{u}_1}{d\widehat{x}} = \frac{d\widehat{u}_2}{d\widehat{x}}}$$

$$= -\frac{\mathrm{Re}^2}{\widehat{\rho}_1 \widehat{\alpha}_1^4}\left[\underbrace{\int_0^1 \widehat{u}_1^2 \frac{d\widehat{u}_1}{d\widehat{x}} d\widehat{x}} - \int_0^1 \frac{d\widehat{u}_1}{d\widehat{x}} d\widehat{x}\right] + \frac{\mathrm{Re}^2}{\widehat{\rho}_2 \widehat{\alpha}_2^4}\left[\int_0^1 \frac{d\widehat{u}_2}{d\widehat{x}} d\widehat{x} - \int_0^1 \widehat{u}_2^2 \frac{d\widehat{u}_2}{d\widehat{x}} d\widehat{x}\right]$$

$$\boxed{\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx}$$

$v = \widehat{u}_1$

$$\int_0^1 \widehat{u}_1^2 \frac{d\widehat{u}_1}{d\widehat{x}} d\widehat{x} = \widehat{u}_1(1)^3 - \widehat{u}_1(0)^3 - \underbrace{\int_0^1 \widehat{u}_1 \frac{d\widehat{u}_1^2}{d\widehat{x}} d\widehat{x}}$$

$u = \widehat{u}_1^2$

$$\boxed{\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}}$$

$$\int_0^1 \widehat{u}_1 \frac{d\widehat{u}_1^2}{d\widehat{x}} d\widehat{x} = \int_0^1 \widehat{u}_1 \left(2\widehat{u}_1 \frac{d\widehat{u}_1}{d\widehat{x}}\right) d\widehat{x} = 2\int_0^1 \widehat{u}_1^2 \frac{d\widehat{u}_1}{d\widehat{x}} d\widehat{x}$$

Calculating component pressure changes: Velocity change (cont.)

$$\left[\int_0^1 \hat{u}_1^2 \frac{d\hat{u}_1}{d\hat{x}} d\hat{x}\right] = -1 - 2\left[\int_0^1 \hat{u}_1^2 \frac{d\hat{u}_1}{d\hat{x}} d\hat{x}\right] \Rightarrow \int_0^1 \hat{u}_1^2 \frac{d\hat{u}_1}{d\hat{x}} d\hat{x} = -\tfrac{1}{3}$$

Similarly,

$$\left[\int_0^1 \hat{u}_2^2 \frac{d\hat{u}_2}{d\hat{x}} d\hat{x}\right] = 1 - 2\left[\int_0^1 \hat{u}_2^2 \frac{d\hat{u}_2}{d\hat{x}} d\hat{x}\right] \Rightarrow \int_0^1 \hat{u}_1^2 \frac{d\hat{u}_1}{d\hat{x}} d\hat{x} = \tfrac{1}{3}$$

Tying up a few more loose ends:

$$\int_0^1 \frac{d\hat{u}_1}{d\hat{x}} d\hat{x} = \hat{u}_1(1) - \hat{u}_1(0) = -1 \qquad \int_0^1 \frac{d\hat{u}_2}{d\hat{x}} d\hat{x} = \hat{u}_2(1) - \hat{u}_2(0) = 1$$

By substitution:

$$\Delta\overline{\hat{P}}_{s,u} = -\frac{\mathrm{Re}^2}{\hat{\rho}_1 \hat{\alpha}_1^4}\left[\left(-\tfrac{1}{3}\right) - (-1)\right] + \frac{\mathrm{Re}^2}{\hat{\rho}_2 \hat{\alpha}_2^4}\left[(1) - \left(\tfrac{1}{3}\right)\right]$$

$$\boxed{\Delta\overline{\hat{P}}_{s,u} = \tfrac{2}{3}\mathrm{Re}^2\left(\frac{1}{\hat{\rho}_2 \hat{\alpha}_2^4} - \frac{1}{\hat{\rho}_1 \hat{\alpha}_1^4}\right)}$$

Summary of component pressure changes:

Inlet channel friction: $\Delta\widehat{P}_1 = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\,\widehat{\alpha}_1^4}\int_0^1 \widehat{u}_1^2\,d\widehat{x}$

Exit channel friction: $\Delta\widehat{P}_2 = \frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\,\widehat{\alpha}_2^4}\int_0^1 \widehat{u}_2^2\,d\widehat{x}$

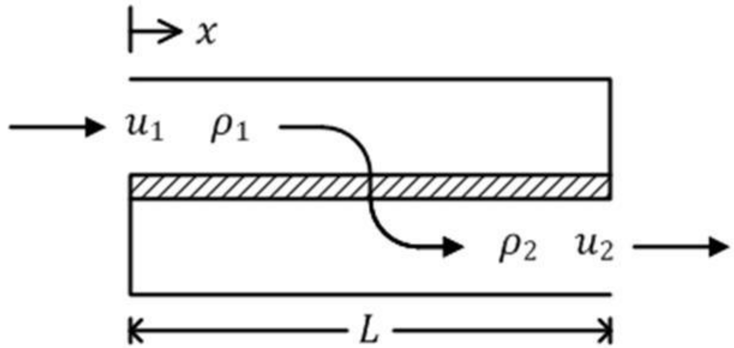
Wall flow: $\Delta\widehat{P}_w = \int_0^1 (\mathcal{K}\mathrm{Re}\,\widehat{u}_w^2 + \mathcal{B}\mathrm{Re}^2\,\widehat{u}_w^3)\,d\widehat{x}$

Velocity change: $\Delta\widehat{P}_u = \frac{2}{3}\mathrm{Re}^2\left(\frac{1}{\widehat{\rho}_2\,\widehat{\alpha}_2^4} - \frac{1}{\widehat{\rho}_1\,\widehat{\alpha}_1^4}\right)$

The total pressure drop is the sum of all the components:

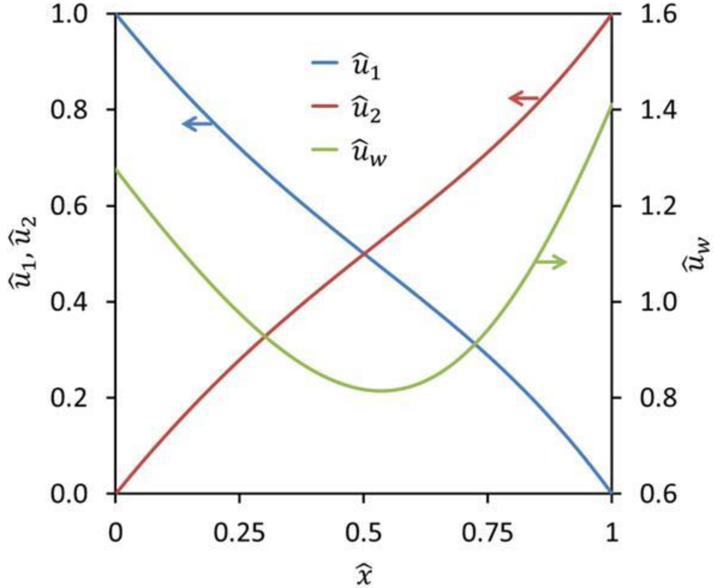
$$\Delta\widehat{P} = \int_0^1 (\mathcal{K}\mathrm{Re}\,\widehat{u}_w^2 + \mathcal{B}\mathrm{Re}^2\,\widehat{u}_w^3)\,d\widehat{x} + \mathcal{F}\mathrm{Re}\int_0^1 \left(\frac{\widehat{u}_1^2}{\widehat{\rho}_1\,\widehat{\alpha}_1^4} + \frac{\widehat{u}_2^2}{\widehat{\rho}_2\,\widehat{\alpha}_2^4}\right)d\widehat{x} + \frac{2}{3}\mathrm{Re}^2\left(\frac{1}{\widehat{\rho}_2\,\widehat{\alpha}_2^4} - \frac{1}{\widehat{\rho}_1\,\widehat{\alpha}_1^4}\right)$$

Example calculation:



Example filter specifications:

$\alpha_1 = 0.85$ mm	$\dot{m} = 0.05$ g/s
$\alpha_2 = 1.15$ mm	$k = 1\ \mu m^2$
$w = 0.3$ mm	$\beta = 500\ /\mu m$
$L = 150$ mm	$T = 1000$ K
$n_w = 3.95$	$P_2(L) = 106.3$ kPa



$$\Delta\hat{P}_1 = 2.7960 \times 10^6$$
$$\Delta\hat{P}_2 = 0.9675 \times 10^6$$
$$\Delta\hat{P}_w = 1.2094 \times 10^6$$
$$\Delta\hat{P}_u = -0.9917 \times 10^6$$
$$\Delta\hat{P}_{Total} = 3.9811 \times 10^6$$

$$\Delta\hat{P}_{Total} = \frac{\mathcal{F}\mathrm{Re}}{\hat{\rho}_1\hat{\alpha}_1^4}\int_0^1 \hat{u}_1 d\hat{x} - \frac{\mathrm{Re}^2}{\hat{\rho}_1\hat{\alpha}_1^4} + \mathcal{K}\mathrm{Re}\hat{u}_w(1) + \mathcal{B}\mathrm{Re}^2\hat{u}_w(1)^2 = 3.9811 \times 10^6$$

$$\Delta\hat{P}_{Total} = \frac{\mathcal{F}\mathrm{Re}}{\hat{\rho}_2\hat{\alpha}_2^4}\int_0^1 \hat{u}_2 d\hat{x} + \frac{\mathrm{Re}^2}{\hat{\rho}_2\hat{\alpha}_2^4} + \mathcal{K}\mathrm{Re}\hat{u}_w(0) + \mathcal{B}\mathrm{Re}^2\hat{u}_w(0)^2 = 3.9811 \times 10^6$$

Assuming uniform wall flow:

$$\widehat{u}_1(\widehat{x}) = 1 - \widehat{x} \qquad \widehat{u}_2(\widehat{x}) = \widehat{x} \qquad \widehat{u}_w(\widehat{x}) = 1$$

$$\Delta\widehat{P} = \int_0^1 (\mathcal{K}\mathrm{Re} + \mathcal{B}\mathrm{Re}^2)d\widehat{x} + \mathcal{F}\mathrm{Re}\int_0^1 \left(\frac{(1-\widehat{x})^2}{\widehat{\rho}_1\widehat{\alpha}_1^4} + \frac{\widehat{x}^2}{\widehat{\rho}_2\widehat{\alpha}_2^4}\right)d\widehat{x} + \tfrac{2}{3}\mathrm{Re}^2\left(\frac{1}{\widehat{\rho}_2\widehat{\alpha}_2^4} - \frac{1}{\widehat{\rho}_1\widehat{\alpha}_1^4}\right)$$

$$\Delta\widehat{P} = \mathcal{K}\mathrm{Re} + \mathcal{B}\mathrm{Re}^2 + \tfrac{1}{3}\frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_1\widehat{\alpha}_1^4} + \tfrac{1}{3}\frac{\mathcal{F}\mathrm{Re}}{\widehat{\rho}_2\widehat{\alpha}_2^4} + \tfrac{2}{3}\mathrm{Re}^2\left(\frac{1}{\widehat{\rho}_2\widehat{\alpha}_2^4} - \frac{1}{\widehat{\rho}_1\widehat{\alpha}_1^4}\right)$$

Neglecting the Forchheimer term, assuming symmetric cells and incompressible flow:

$$\mathcal{B} = 0 \qquad \widehat{\alpha}_1 = \widehat{\alpha}_2 = 1 \qquad \widehat{\rho}_1 = \widehat{\rho}_2 = 1$$

$$\Delta\widehat{P} = \mathcal{K}\mathrm{Re} + \tfrac{2}{3}\mathcal{F}\mathrm{Re}$$

And in dimensional form, with $n_w = 4$:

$$\Delta P = \frac{\mu u_0 \alpha w}{4kL} + \frac{2}{3}\frac{\mu F u_0 L}{\alpha^2}$$

Conclusions:

- Integral expressions for component pressure changes have been developed and presented .
- Component pressure changes vary depending on wall crossing location.
- Overall pressure changes can be computed as the mass average over all possible gas paths.
- The expressions reduce to previously published forms when simplifying assumptions are made.
- The $\frac{2}{3}$ comes from integration of the inlet and outlet channel velocities *squared*.
- Example calculations were performed, and produced results which are self-consistent.