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Analytical Models for Particulate Filter Backpressure Prediction

Tim Watling

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Background: Analytic Filter Models

- 1-dimensional models of particulate filters typically involve numeric solution of mass, momentum and energy balance equations

- However, for isothermal conditions and uniform soot- and ash-deposits, the mass and momentum balance equations can be solved analytically
- Advantages of analytic models:
  - Faster solution
  - Identification of general trends easier

- A number of analytic filter models available in the literature
  - Two types:
    - Full/detailed analytic model: Aim to include all the features of an isothermal numeric model
    - "Simple analytic model": Simpler equation at expense of lower range of applicability

Outline

- Simple analytic filter model
  - History of previous simple analytic filter models
    - These come from Konstandopoulos, Johnson and others (K&J type models)
  - Derive a series of simplified analytic models in an attempt to derive K&J-type model
  - Comparison of simple analytic models with numeric models

- ~~Full analytic model~~
  - ~~Derive model with more features than previous models, e.g. octo-square asymmetric filters~~
  - ~~Comparison with numeric model~~

Simple Analytic Filter Model

History of Simple Analytic Filter Models

None of these equations has actually been derived

- d: Soot-free channel width
- d_i: Soot-loaded channel width
- F: Viscous loss coefficient
- L: Length of filter
- k_W: Filter wall permeability
- k_S: Soot cake permeability
- $\dot{m}_C$: Mass flow into channel
- w_W: Thickness of filter wall
- μ: Viscosity
- ρ: Gas density

- **Generation 1**: Konstandopoulos & Johnson, SAE 890405, 1989
  - $\Delta P = \frac{\mu \dot{m}_C}{d^2 \rho}\left[\frac{1.00456 d w_W}{4 k_W L} + \frac{FL}{1.484 d^2}\right]$
  - Equation not derived, but obtained by curve fitting results of more detailed model
  - For soot-free, symmetric filters
- **Generation 2**: Konstandopoulos et al., SAE 1999-01-0468, 1999
  - $\Delta P = \frac{\mu \dot{m}_C}{d^2 \rho}\left[\frac{d w_W}{4 k_W L} + \frac{2FL}{3 d^2}\right]$
  - Same equation, but constants rounded
- **Generation 3**: Konstandopoulos et al., SAE 2000-01-1016, 2000
  - $\Delta P = \frac{\mu \dot{m}_C}{\rho}\left[\frac{1}{4L}\left(\frac{\mathrm{w_W}}{\mathrm{dk_W}} + \frac{1}{2\mathrm{k_S}}\ln\left(\frac{\mathrm{d}}{\mathrm{d_i}}\right)\right) + \frac{FL}{3}\left(\frac{1}{d_i^4} + \frac{1}{d^4}\right)\right]$
  - Extension to soot-loaded filters. Here inlet and outlet channels have a different width
  - Not derived, but argued that can separate term for channel pressure drop into terms for each channel

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Simple Analytic Filter Models: Questions

- Can these simple equations be derived from first principles?
- For what range of conditions are they valid?
  - e.g. what flow rates, wall permeability, etc.
- Form of equation suggests that terms for along-channel and across-wall & across-soot pressure loss are separable. Is this true?

- Might expect equation for octo-square asymmetric filter to be:
  - $\Delta P = \frac{\mu \dot{m}_C}{\rho}\left[\frac{1}{4L}\left(\frac{1}{[2-\sqrt{2}]k_W}\ln\left(\frac{\mathrm{d_o}+[2-\sqrt{2}]w_W}{d_o}\right)+\frac{\mathrm{d}}{\mathrm{pk_S}}\ln\left(\frac{A}{\mathrm{A_i}}\right)\right)+\frac{L}{3}\left(\frac{F_i}{A_i^2}+\frac{F_o}{d_o^4}\right)\right]$
  - Equation for pressure drop across wall and soot cake from SAE 2018-01-0951

A :	Cross-section area of soot-free inlet channel
A_i:	Cross-sectional area of soot-loaded inlet channel
d_o:	Width of outlet channel
F_i, F_o:	Viscous loss coefficient for inlet and outlet channels

Pakko (CLEERS 2016) previously presented a derivation of K&J model

Derivation 1: “Low Inertia Model”

Balance equations & Darcy’s law

- Consider an isothermal filter with a soot cake of uniform thickness
- Assume relatively low flow rates: incompressible flow, neglect inertial terms
- But allow for channels to be different so works for asymmetric and soot-loaded filters

- Mass balance: $A_i \frac{\partial \phi_i}{\partial z} = -4\psi_W \qquad d_o^2 \frac{\partial \phi_o}{\partial z} = 4\psi_W$
- Momentum balance: $\frac{\partial P_i}{\partial z} = -\frac{F_i \mu \phi_i}{A_i \rho} \qquad \frac{\partial P_o}{\partial z} = -\frac{F_o \mu \phi_o}{d_o^2 \rho}$
- Pressure drop across filter wall and soot cake:

$$P_i - P_o = \frac{K}{\rho} \mu \psi_W$$

  - where K is a constant with wall & soot permeabilities and geometric parameters

A_i: Cross-sectional area of soot-loaded inlet channel
d_o: Width of outlet channel
F_i, F_o: Viscous loss coeffn for inlet & outlet channels
K : Const for soot & wall geometry & permeabilities
P_i, P_o: Pressure in inlet & outlet channels
z : Axial coordinate
μ : Gas viscosity
ρ : Gas density
ϕ_i: Mass flux of gas along inlet channel, ρV
ϕ_o: Mass flux of gas along outlet channel, ρV
ψ_W: Mass flow through wall per unit length of wall

Derivation 1: "Low Inertia Model"

Solution for flow rate in inlet channel

- Derivation follows same line as detailed analytic model of SAE 890405
- Subtract one momentum balance from the other:
  - $\frac{\partial(P_i - P_o)}{\partial z} = -\frac{\mu}{\rho}\left[\frac{F_i}{A_i}\phi_i + \frac{F_o}{d_o^2}\phi_o\right]$
- Substitute in Darcy's law:
  - $K\frac{\partial\psi_W}{\partial z} = -\left[\frac{F_i}{A_i}\phi_i + \frac{F_o}{d_o^2}\phi_o\right]$
- Substitute in mass balance:
  - $\frac{A_i K}{4}\frac{\partial^2\phi_i}{\partial z^2} = \left(\frac{F_i}{A_i} + \frac{A_i F_o}{d_o^4}\right)\phi_i - \frac{F_o \dot{m}_C}{d_o^4}$
- Solve this second order differential equation and apply the boundary conditions:
  - $\frac{A_i\phi_i}{\dot{m}_C} = C + \frac{[1-C]\sinh\alpha[L/2-z] - C\sinh\alpha[L/2+z]}{\sinh\alpha L}$

$$C = \frac{A_i^2 F_o}{d_o^4 F_i + A_i^2 F_o}$$

$$\alpha = \sqrt{\frac{4}{K}\left(\frac{F_i}{A_i^2} + \frac{F_o}{d_o^4}\right)}$$

Domain of filter: − L/2≤z≤L/2

Derivation 1: “Low Inertia Model”

Solution for other variables: outlet & across-wall flow, pressure in channels, etc.

- Solution for inlet channel flow:
  - $\frac{A_i \phi_i}{\dot{m}_C} = C + \frac{[1-C]\sinh \alpha[L/2-z] - C \sinh \alpha[L/2+z]}{\sinh \alpha L}$
- Mass balance gives outlet channel flow and flow though wall from inlet channel flow:
  - $\frac{d_o^2 \phi_o}{\dot{m}_C} = 1 - \mathrm{C} - \frac{[1-C]\sinh \alpha[L/2-z] - C \sinh \alpha[L/2+z]}{\sinh \alpha L}$
  - $\psi_W = \frac{\dot{m}_C \alpha([1-C]\cosh \alpha[L/2-z] + C \cosh \alpha[L/2+z])}{4 \sinh \alpha L}$
- Darcy’s law gives pressure drop across wall & soot cake from through-wall flow rate:
  - $P_i - P_o = \frac{K \mu \dot{m}_C \alpha([1-\mathrm{C}]\cosh \alpha[L/2-z] + C \cosh \alpha[L/2+z])}{4\rho \sinh \alpha L}$
- Substitution of inlet or outlet channel flow into momentum balance & solving gives P:
  - $P_i = -\frac{F_i \mu \dot{m}_C}{A_i^2 \rho}\left[Cz - \frac{[1-C]\cosh \alpha[L/2-z] + C \cosh \alpha[L/2+z]}{\alpha \sinh \alpha L}\right] + D$
  - $P_o = -\frac{F_o \mu \dot{m}_C}{d_o^4 \rho}\left[[1-C]z + \frac{[1-C]\cosh \alpha[L/2-z] + C \cosh \alpha[L/2+z]}{\alpha \sinh \alpha L}\right] + E$

D & E are arbitrary constants

Derivation 1: “Low Inertia Model”

Pressure drop across filter

- There are several contributions to backpressure:
  - Pressure drop along inlet channel
  - Pressure drop across wall and soot cake
  - Pressure drop along outlet channel
- Exact combination of these for a gas molecule depends on where it crosses the wall
- Pressure drop across filter for gas molecule crossing the wall at z_W is given by
  - $\Delta P = \left(P_i|_{z=-L/2} - P_i|_{z_W}\right) + \left(P_i|_{z_W} - P_o|_{z_W}\right) + \left(P_o|_{z_W} - P_o|_{z=L/2}\right)$
- Substituting into this gives:
  - $\Delta \mathrm{P} = \frac{\mu \dot{m}_C}{\rho} \frac{1}{d_o^4 F_i + A_i^2 F_o^2}\left[F_i F_o L + \frac{1}{\alpha \sinh \alpha L}\left(\frac{d_o^8 F_i^2 + A_i^4 F_o^2}{A_i^2 d_o^4} \cosh \alpha L + 2 F_i F_o\right)\right]$
  - Pressure drop across filter does not depend on where gas crosses the wall, as it should

NB Pressure drop due to entrance & exit effects not included, but could be

Derivation 1: “Low Inertia Model”

Compare with Konstandopoulos & Johnson-Type model

- “Low-inertia” solution just derived looks nothing like Konstandopoulos & Johnson-type equation
  - Low-inertia: $\Delta \mathrm{P} = \frac{\mu \dot{m}_C}{\rho} \frac{1}{d_o^4 F_i + A_i^2 F_o^2} \left[F_i F_o L + \frac{1}{\alpha \sinh \alpha L} \left(\frac{d_o^8 F_i^2 + A_i^4 F_o^2}{A_i^2 d_o^4} \cosh \alpha L + 2 F_i F_o \right) \right]$
  - K&J-type: $\Delta P = \frac{\mu \dot{m}_C}{\rho} \left[\frac{L}{3} \left(\frac{F_i}{A_i^2} + \frac{F_o}{d_o^4} \right) + \frac{K}{4L} \right]$

$$\alpha = \sqrt{\frac{4}{K} \left(\frac{F_i}{A_i^2} + \frac{F_o}{d_o^4} \right)}$$

- Key difference is that terms for pressure drop in inlet and outlet channels and across wall are combined in low-inertia solution but completely independent in Konstandopoulos & Johnson-type equation

Derivation 2: "Low α, Low Inertia Model"

$$\alpha = \sqrt{\frac{4}{K}\left(\frac{F_i}{A_i^2} + \frac{F_o}{d_o^4}\right)}$$

- Another attempt at deriving the K&J model…
- Take low-inertia model and assume that α is small
  - Corresponds to pressure drop across wall and soot cake being large compared with along channel pressure drop
- Remember: $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,\ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
  - Therefore, assume $\sinh x = x$ and $\cosh x = 1$ in low-inertia model
- This gives:
  - $\frac{A_i\phi_i}{\dot{m}_C} = \frac{1}{2} - \frac{z}{L}$ $\quad \frac{d_o^2\phi_o}{\dot{m}_C} = \frac{1}{2} + \frac{z}{L}$ $\quad \psi_W = \frac{\dot{m}_C}{4L}$
  - $P_i = -\frac{F_iF_o\mu\dot{m}_C}{\rho(d_o^4F_i + A_i^2F_o)}z + D'$ $\quad P_o = -\frac{F_iF_o\mu\dot{m}_C}{\rho(d_o^4F_i + A_i^2F_o)}z + E'$
  - $\Delta P = \frac{\mu\dot{m}_C}{\rho}\left[\frac{F_iF_oL}{d_o^4F_i + A_i^2F_o} + \frac{K}{4L}\right]$ Still different from K&J model

i.e. constant through-wall velocity along channels; inlet and outlet channel velocities and pressure vary linearly along filter

Derivation 3: Assume Const Wall Velocity & Apply Momentum Balance

- Next attempt at deriving the K&J model…
  - Assume constant wall-flow along the length of the filter, as predicted by "low-α" model
  - Substitute this into momentum balances and evaluate
- This gives:
  - $P_i = -\frac{F_i \mu \dot{m}_C}{2Ai^2 \rho}\left[z - \frac{z^2}{L}\right] + D''$ $\qquad$ $P_o = -\frac{F_o \mu \dot{m}_C}{2d_o^4 \rho}\left[z + \frac{z^2}{L}\right] + E''$
- Need different approach for calculating total pressure drop (as model doesn't "add up")
  - Hence, $\Delta P = \overline{\Delta P_i} + \overline{\Delta P_W} + \overline{\Delta P_o}$
    - where $\overline{\Delta P_i} = \frac{1}{L}\int_{-L/2}^{L/2}\left(P_i|_{z=-L/2} - P_i|_z\right)dz = \frac{F_i \mu \dot{m}_C L}{3Ai^2 \rho}$ $\qquad$ $\overline{\Delta P_o} = \frac{1}{L}\int_{-L/2}^{L/2}\left(P_o|_z - P_o|_{z=L/2}\right)dz = \frac{F_o \mu \dot{m}_C L}{3d_o^4 \rho}$
  - This gives:
    - $\Delta P = \frac{\mu \dot{m}_C}{\rho}\left[\frac{L}{3}\left(\frac{F_i}{A_i^2} + \frac{F_o}{d_o^4}\right) + \frac{K}{4L}\right]$ $\qquad$ Finally, we have the K&J model!

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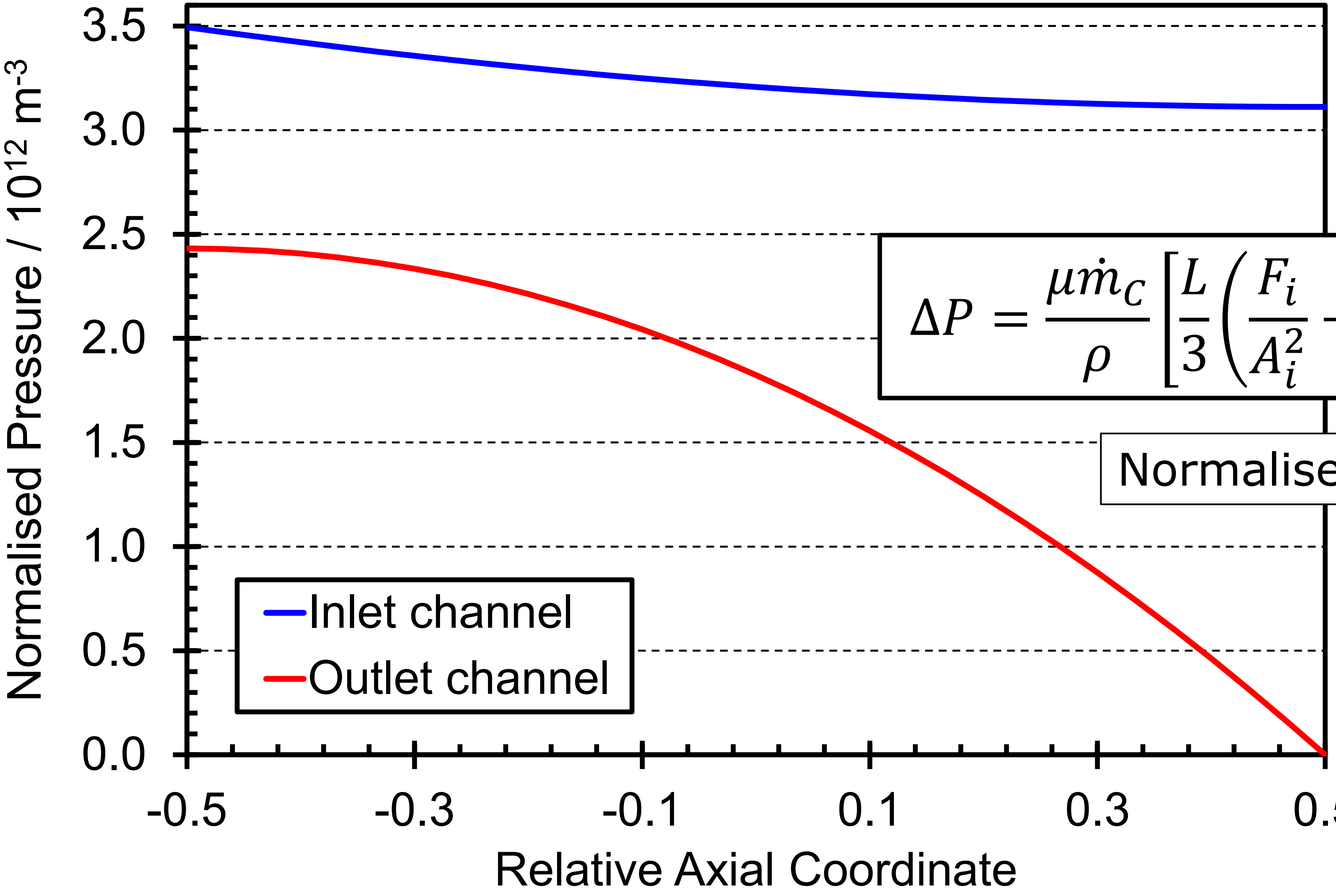
Konstandopoulos-Johnson Type Model: Pressure Profile Prediction

118.4×152.4 mm, 300/8, d/d_o=1.6 octo-square asymmetric, $k_w=2\times10^{-13}$ m², 20°C



$$\Delta P = \frac{\mu \dot{m}_C}{\rho}\left[\frac{L}{3}\left(\frac{F_i}{A_i^2}+\frac{F_o}{d_o^4}\right)+\frac{K}{4L}\right]$$

Normalised P=$\rho P/\mu \dot{m}_C$

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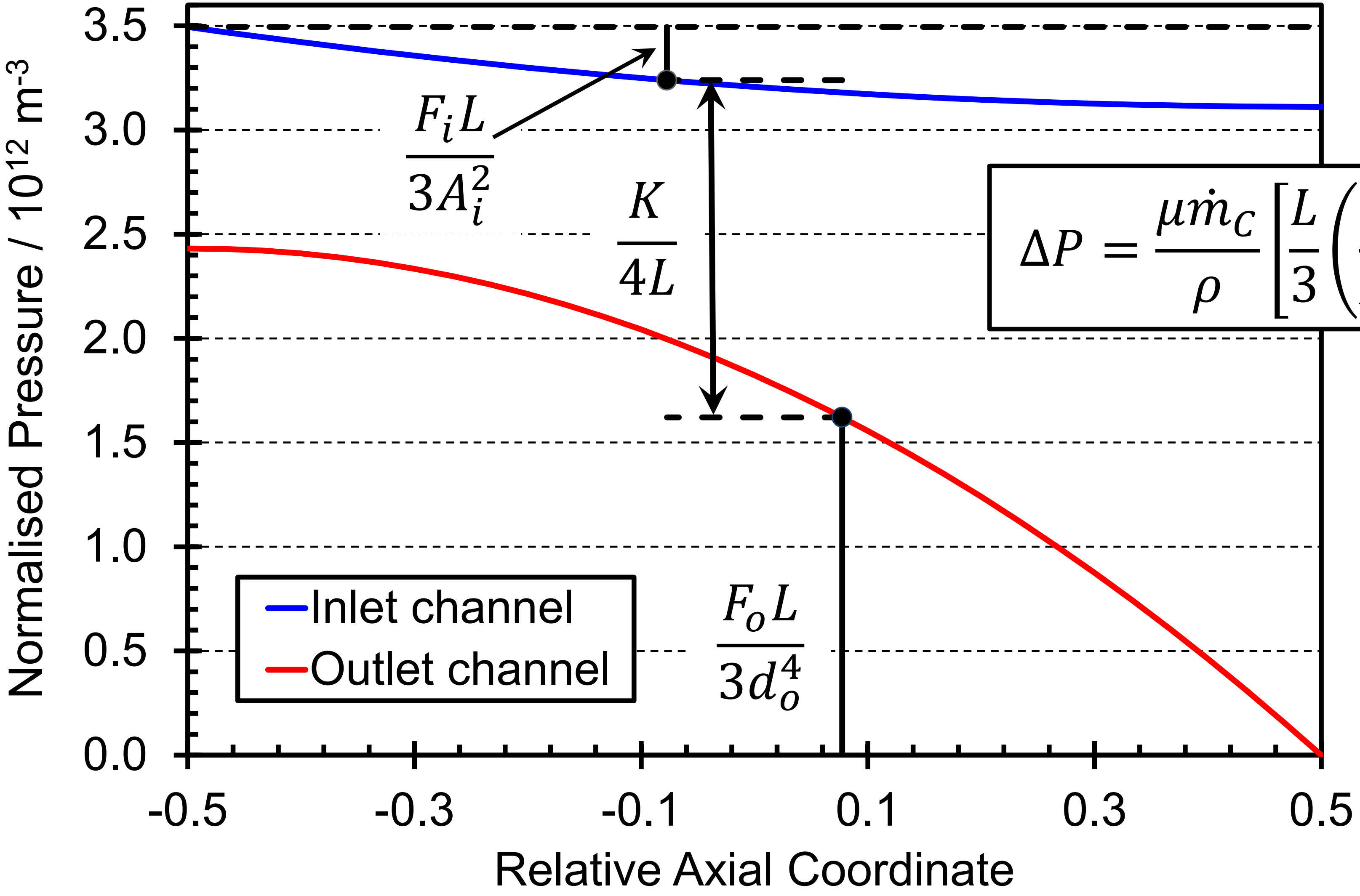
Konstandopoulos-Johnson Type Model: Pressure Profile Prediction

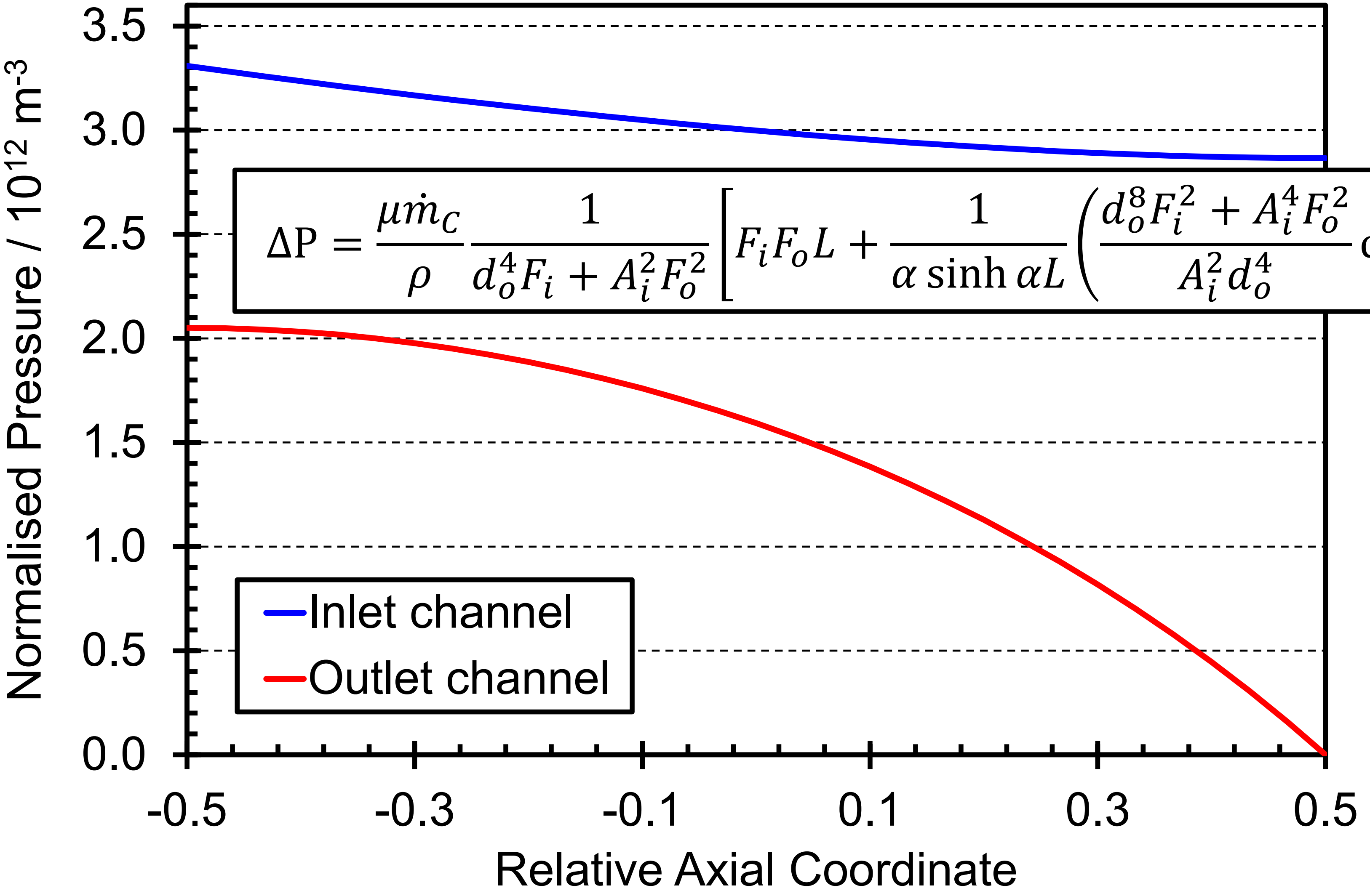
Terms in pressure drop equation not entirely consistent with pressure profiles



$$\Delta P = \frac{\mu \dot{m}_C}{\rho}\left[\frac{L}{3}\left(\frac{F_i}{A_i^2} + \frac{F_o}{d_o^4}\right) + \frac{K}{4L}\right]$$

Low Inertia Model: Pressure Profile Prediction

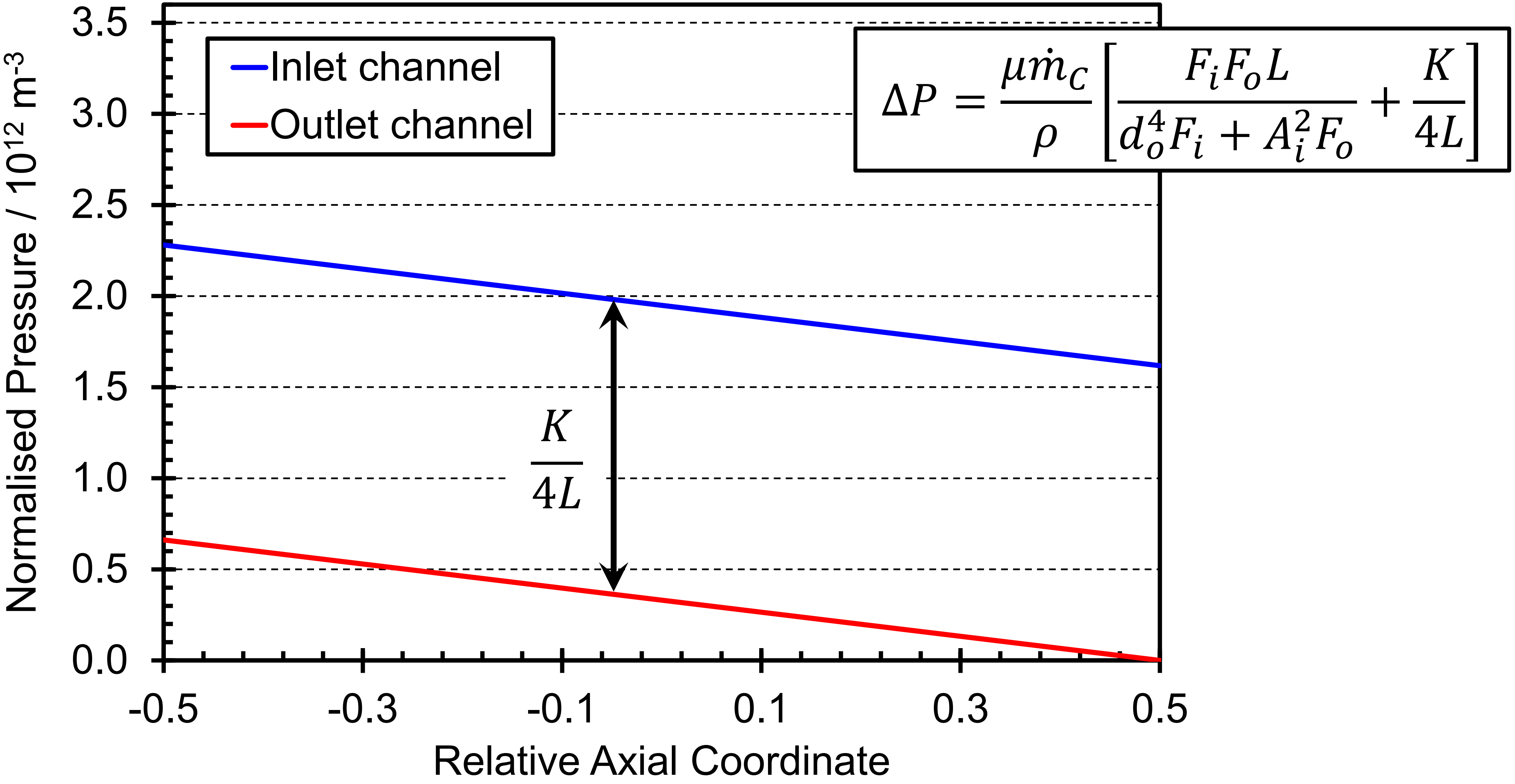
118.4×152.4 mm, 300/8, d/d_o=1.6 octo-square asymmetric, $k_w=2\times10^{-13}$ m², 20°C



$$\Delta P = \frac{\mu \dot{m}_C}{\rho} \frac{1}{d_o^4 F_i + A_i^2 F_o^2} \left[F_i F_o L + \frac{1}{\alpha \sinh \alpha L} \left(\frac{d_o^8 F_i^2 + A_i^4 F_o^2}{A_i^2 d_o^4} \cosh \alpha L + 2 F_i F_o \right) \right]$$

Low α Model: Pressure Profile Prediction

118.4×152.4 mm, 300/8, d/d_o=1.6 octo-square asymmetric, k_W=2×10^{-13} m^2, 20°C



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Comparison of Numeric and Analytic Models

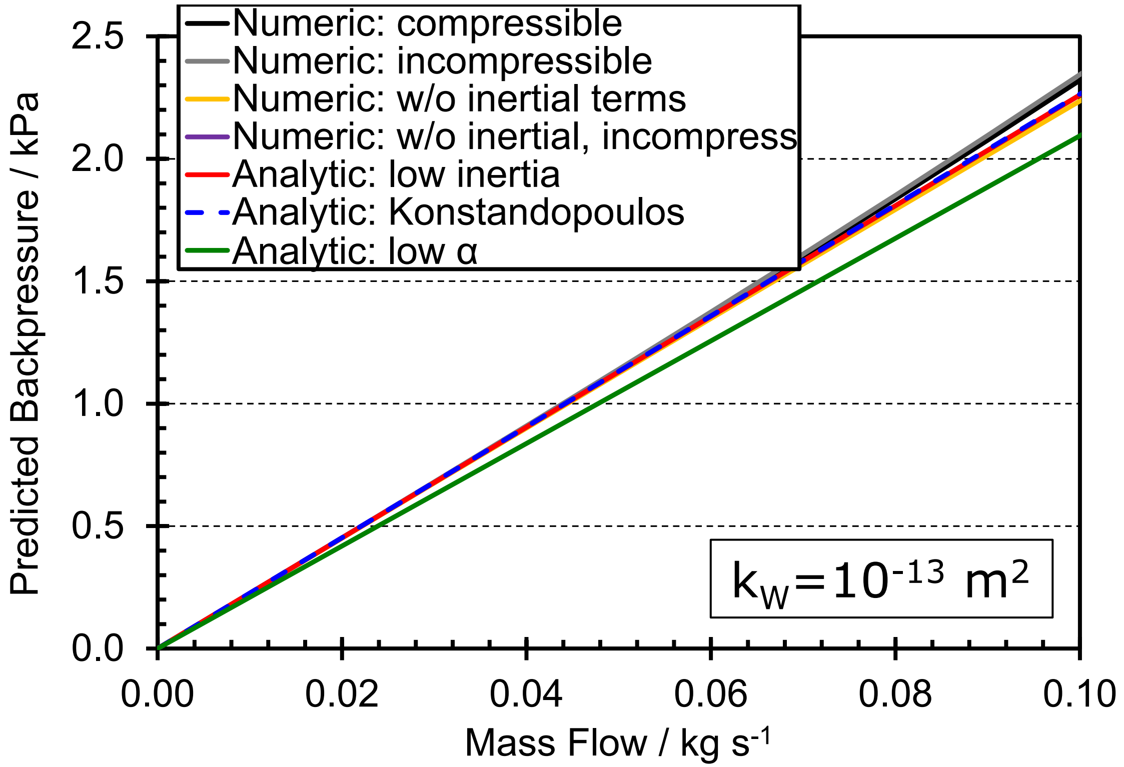
- Compare the following models:
  - Numeric:
    - Model with and without inertial terms
    - Compressible and incompressible flow

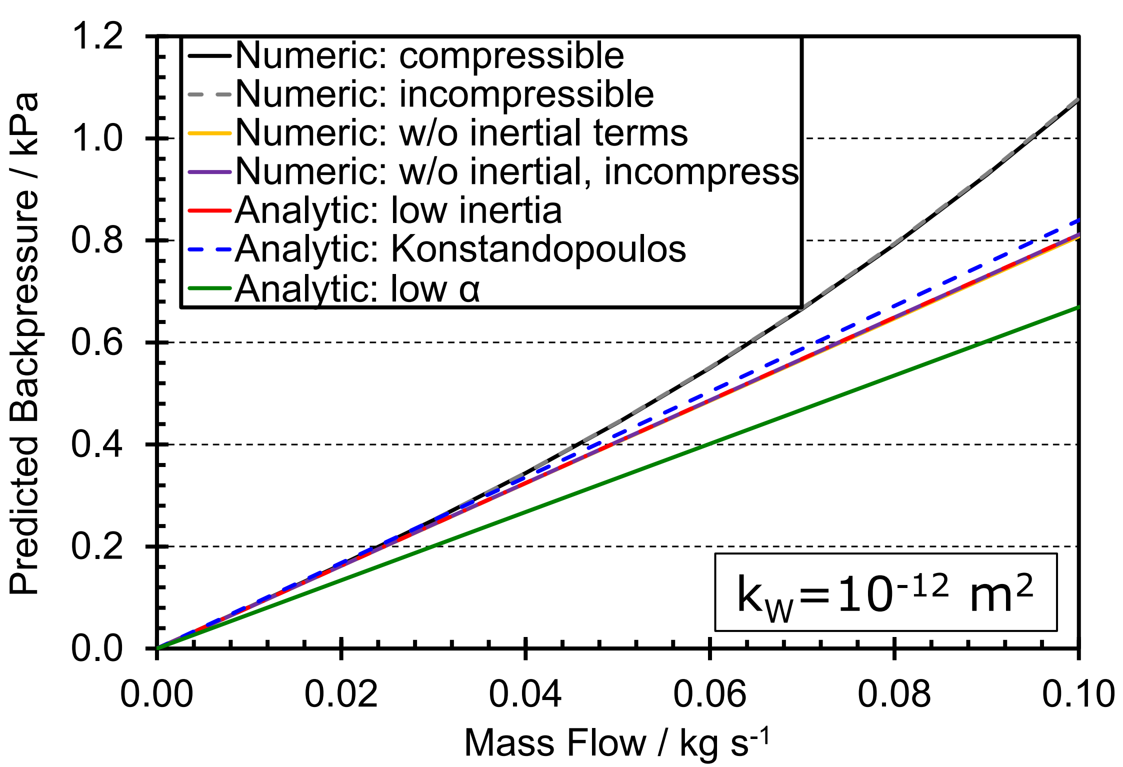
    } 4 models
  - Analytic:
    - Low inertia model
    - Low-ɑ model
    - K&J-type model

System:
118.4×152.4mm
300/8
Symmetric (unless otherwise stated)
20°C

- Two types of comparison:
  - Compare numeric and analytic models when vary flow rate for fixed configuration
  - Compare analytic models when vary i) soot loading and ii) channel width ratio of asymmetric

Model Comparison: Effect of Flow Rate and Wall Permeability

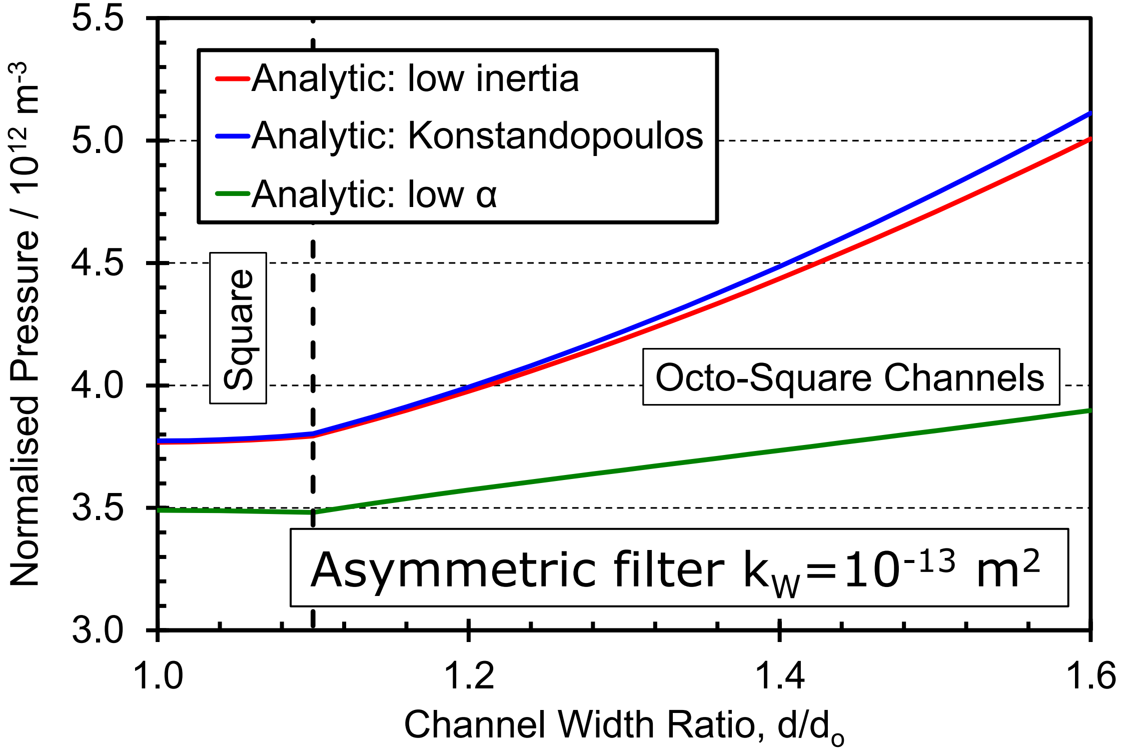


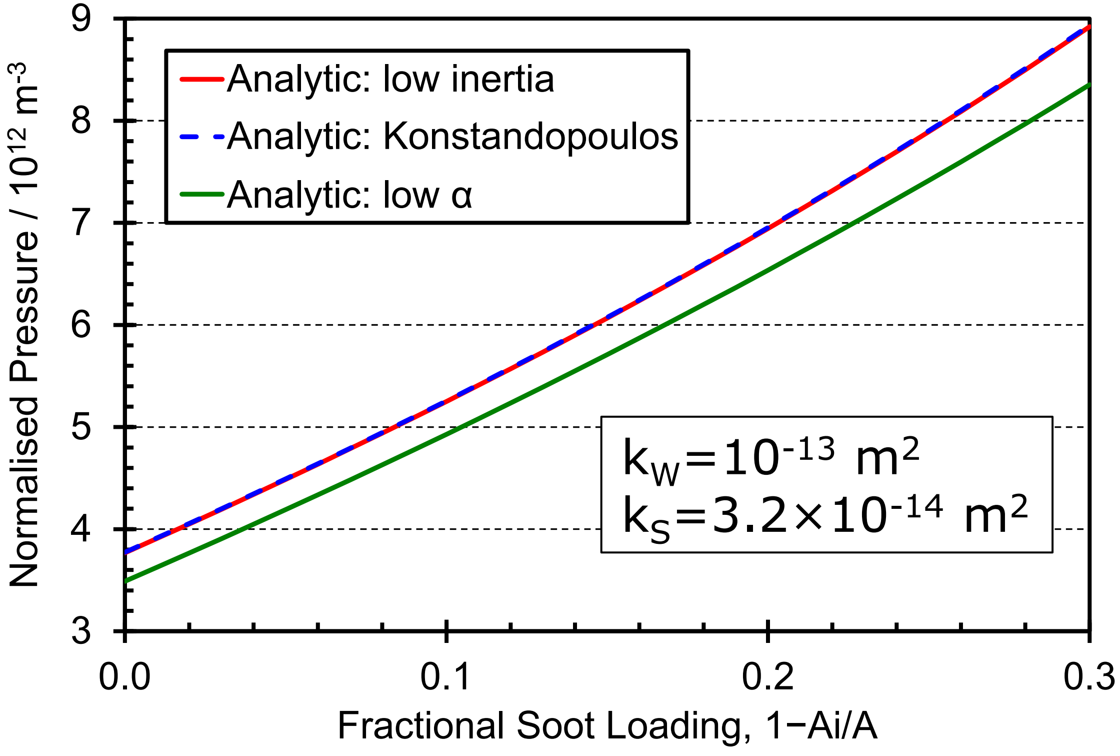


- Inertial terms can be neglected for lower flows and lower permeability
- Low inertia model agrees perfectly with incompressible, numeric model w/o inertial terms
- Konstandopoulos & Johnson model agrees well with low-inertia, incompressible flow models
- Low α model under predicts compared to other models

Model Comparison: Effect of Channel Width Ratio and Soot Loading

Normalised $\Delta P = \rho \Delta P / \mu \dot{m}_C$



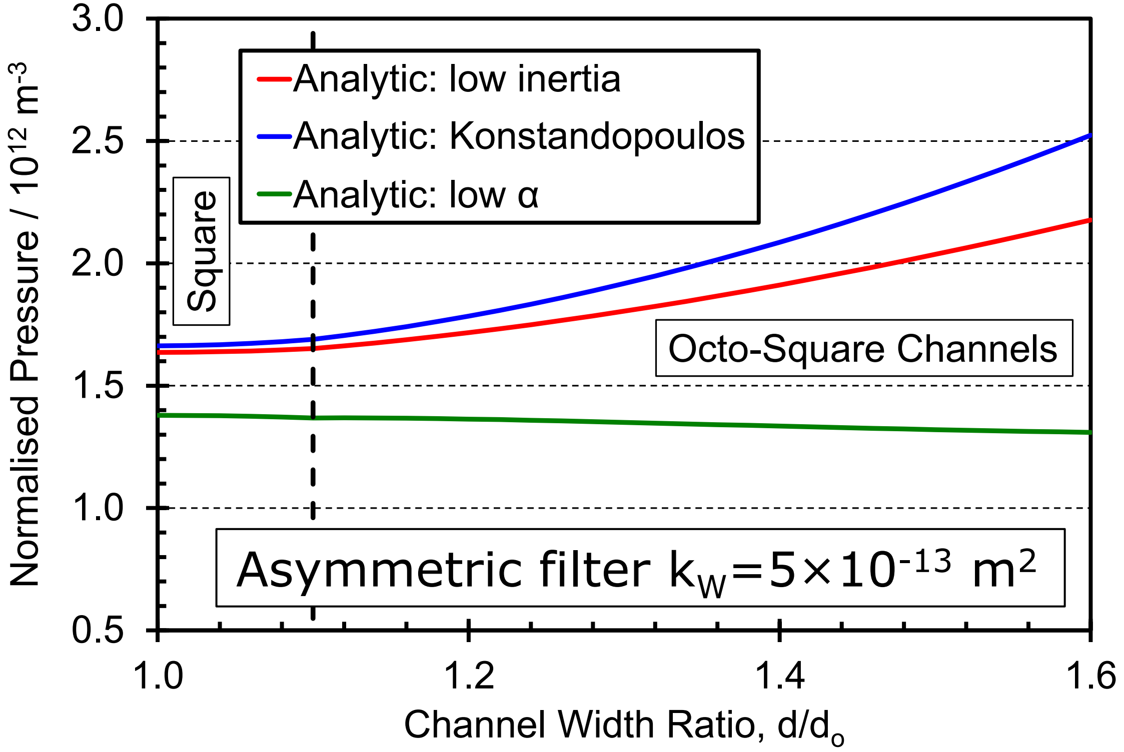


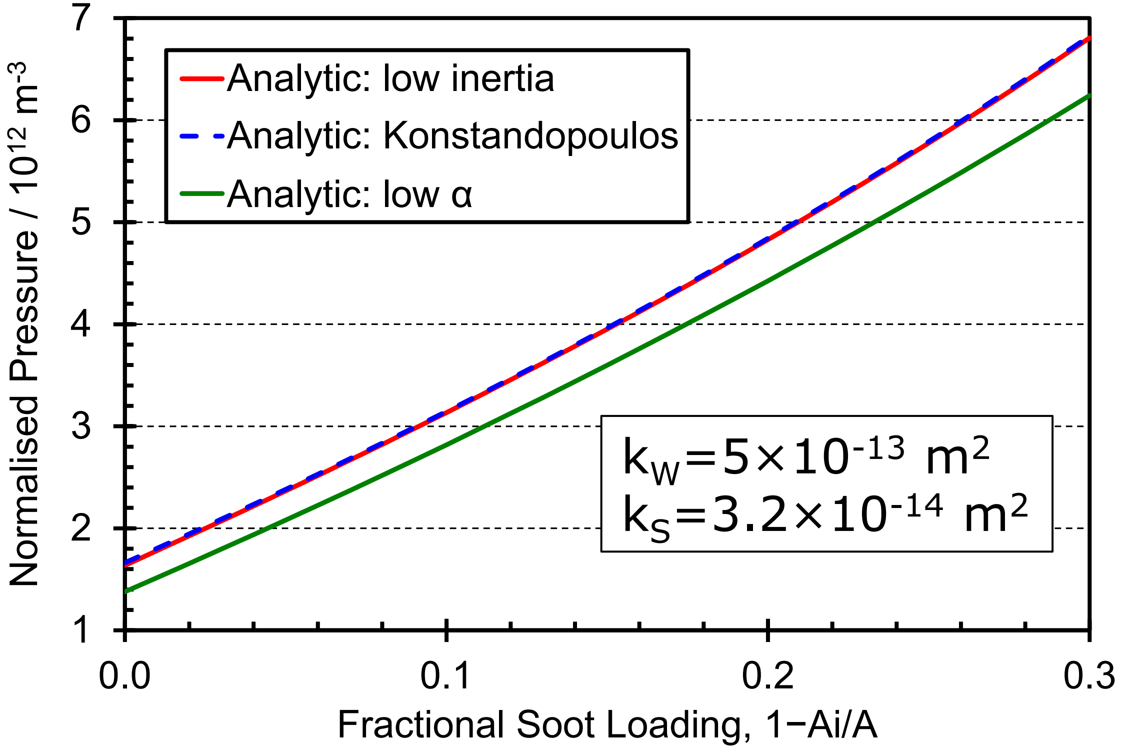
- When soot load, good agreement with low-inertia and K&J – helped by high p-drop across soot
- With octo-square asymmetric, difference in pressure drop prediction between low-inertia and K&J increases as filter becomes more asymmetric (but small difference)
- Low-α model underpredicts compared to other models

Model Comparison: Effect of Channel Width Ratio and Soot Loading

Increase wall permeability

Normalised ΔP= $\rho\Delta P/\mu\dot{m}_C$





- When soot load, good agreement with low-inertia and K&J – helped by high p-drop across soot
- With octo-square asymmetric, difference in pressure drop prediction between low-inertia and K&J increases as filter becomes more asymmetric – greater difference with higher permeability
- Low-α model underpredicts compared to other models

When is Konstandopoulos & Johnson-Type Model Valid?

- Konstandopoulos & Johnson-type model valid when:
  - Inertial terms can be neglected
  - Constant wall-flow along filter

- For inertial terms to negligible need:
  - Relatively low space velocity
  - Low wall & soot permeability, thick soot cake and filter wall
    - If pressure drop across wall & soot cake large, inertial contributions small in comparison

- For constant wall-flow along filter need:
  - Low wall & soot permeability, thick soot cake and filter wall
  - Negligible inertial effects

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Conclusions

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Conclusions

- A series of analytic models for filter backpressure prediction have been developed

- Konstandopoulou & Johnson-type models:
  - Work well, despite simplicity of equation and less than rigorous derivation
    - Provided flow rate relatively low & pressure drop across wall & soot cake relatively high
  - Has been extended to asymmetric filters with different channel cross sections

- Predictions of low inertia analytic model match low-inertia, incompressible numeric model